Sinking Point

Dynamic precision tracking for floating-point

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IEEE 754 floating-point

- Fast, portable, implemented in hardware
- Every operation is completely specified
- Often behavior is similar enough to real numbers
  - But sometimes it isn't!
  - Can be difficult to reason about

A toy example

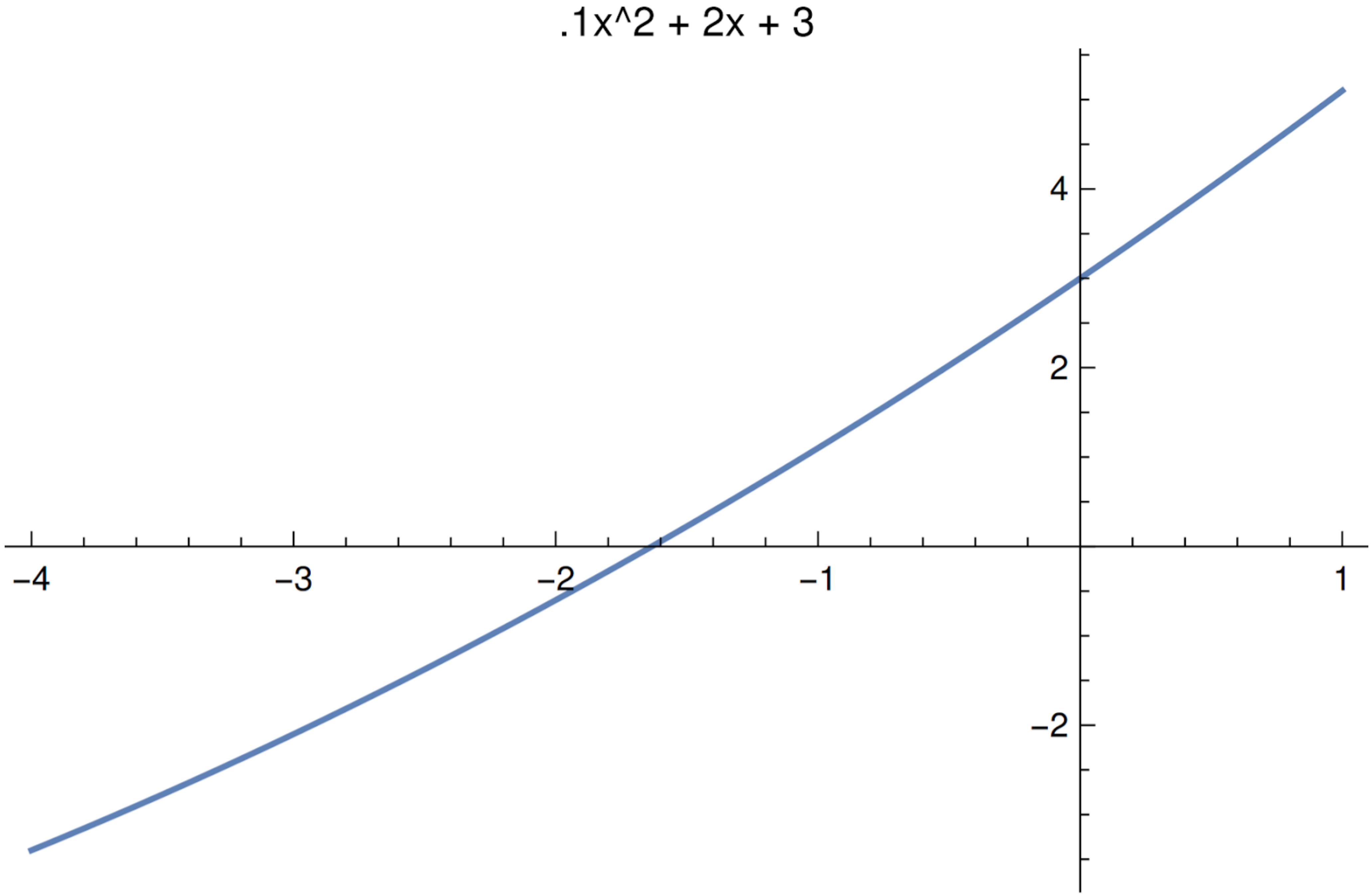
```
$ python
Python 3.6.5 |Anaconda, Inc.|
(default, Apr 29 2018, 16:14:56)
[GCC 7.2.0] on linux
>>> import math
>>> math.pi + 1e16 - 1e16
4.0
>>>
```

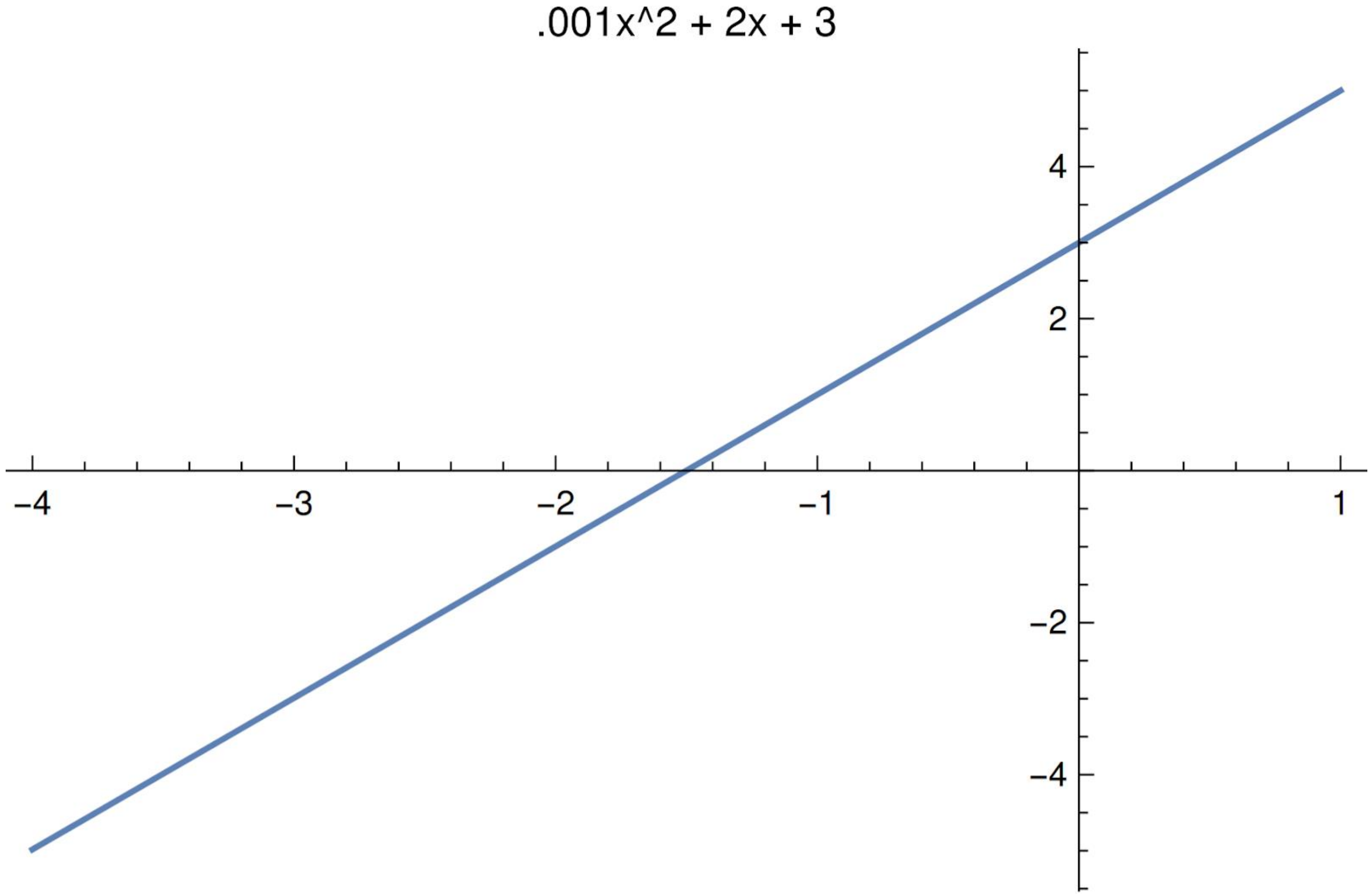
The quadratic formula

$$ax^2 + bx + c = 0$$

Suppose:

- $b = 2$
- $c = 3$
- a is very small


.1x^2 + 2x + 3
4
2
−2
−4
−3
−2
−1
1


.001x^2 + 2x + 3
4
2
−2
−4
−4
−3
−2
−1
1

$$ax^2 + bx + c = 0 \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

a	*b*	*c*	*x* (IEEE 754 double)
.1	2	3	-1.6333997346592444
.001	2	3	-1.5011266906707066
1e-9	2	3	-1.500000013088254
1e-15	2	3	-1.5543122344752189
1e-16	2	3	-2.2204460492503131
1e-17	2	3	0

This behavior is “correct”

- IEEE 754 floating-point is fully specified
  - Just doing what the standard says
- Up to the programmer to determine which bits are meaningful
  - Write code carefully (like libm)
  - Code analysis tools (FPTaylor, Herbie)
  - Track accuracy dynamically throughout the computation.

Sinking-point

- IEEE 754 has a fixed amount of precision
  - Based on available storage, not the properties of the computation
- Sinking-point allows precision to vary
  - Dynamically drop bits from the precision of the significand
  - Compact to store, fast to compute
    - No complex interval computations
  - Still an approximation
    - NOT a sound guarantee
    - NOT a replacement for interval analysis

Printing sinking-point numbers

- Unlike IEEE 754, sinking-point can represent numbers with the same value but different amounts of precision
  - They need to print differently and uniquely

    1.5 (how many bits is that?)

    1.[3-7] (1.5 with 2 bits of precision)

    1.[4991-5009] (1.5 with 10 bits of precision)

- Each “interval” *uniquely* identifies a number
  - NOT interval arithmetic intervals

$$ax^2 + bx + c = 0 \qquad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

a	*b*	*c*	*x* (IEEE 754 double)	*x* (exact)	*x* (sinking-point)
.1	2	3	-1.6333997346592444	-1.6333997346592446	-1.63339973465924[0-8]
.001	2	3	-1.5011266906707066	-1.5011266906707219	-1.50112669067[68-78]
1e-9	2	3	-1.5000000013088254	-1.5000000011250001	-1.[49999995-50000005]
1e-15	2	3	-1.5543122344752189	-1.5000000000000011	-1.[44-56]
1e-16	2	3	-2.2204460492503131	-1.5000000000000002	-[1.8-2.5]
1e-17	2	3	0	-1.5000000000000000	[-1.-+1.]

$$ax^2 + bx + c = 0 \qquad x = \frac{1}{(\sqrt{b^2 - 4ac} + b)(\frac{-1}{2c})}$$

a	*b*	*c*	*x* (IEEE 754 double)	*x* (exact)	*x* (sinking-point)
.1	2	3	-1.6333997346592446	-1.6333997346592446	-1.633399734659244[5-7]
.001	2	3	-1.5011266906707219	-1.5011266906707219	-1.5011266906707072[18-20]
1e-9	2	3	-1.5000000011250001	-1.5000000011250001	-1.50000000112500[0-2]
1e-15	2	3	-1.5000000000000013	-1.5000000000000011	-1.500000000000001[3-4]
1e-16	2	3	-1.5000000000000004	-1.5000000000000002	-1.500000000000000[4-5]
1e-17	2	3	-1.5	-1.5000000000000000	-1.[4999999999999999 -5000000000000001]

Sinking-point addition

- Sinking-point stores extra information
  - (value, *prec*)
- $(v_1, prec_1) + (v_2, prec_2) = (v_3, prec_3)$

5.25 + 4.015625

p = 5 **n** = -3

```
  0b101.01????
+ 0b100.000001
--------------
```

What information should *prec* hold?

- prec = (**p**, **n**)
- **p** is the “bitwidth” of the number
- **n** is the “least known bit”

Sinking-point addition

- Sinking-point stores extra information
    - (value, p, n)
- $(v_1, p_1, n_1) + (v_2, p_2, n_2) = (v_3, p_3, n_3)$

$(v_1 = 5.25, p_1 = 5, n_1 = -3) + (v_2 = 4.015625, p_2 = 9, n_2 = -7)$

```
   0b101.01????
 + 0b100.000001
 --------------
  0b1001.010001
  0b1001.01
```

$n_3 = -3 \quad = \max(n_1, n_2)$

$v_3 = 9.25 = \mathrm{round}(v_1+v_2, n_3)$

$p_3 = 6$

Subtraction

- Addition, but with opposite signs
- $(v_1, p_1, n_1) - (v_2, p_2, n_2) = (v_3, p_3, n_3)$

$(v_1 = 5.25, p_1 = 5, n_1 = -3) - (v_2 = 4.015625, p_2 = 9, n_2 = -7)$

```
   0b101.01????
 - 0b100.000001
 --------------
     0b1.001111
     0b1.01
```

$n_3 = -3 \quad = \max(n_1, n_2)$

$v_3 = 1.25 = \mathrm{round}(v_1 - v_2, n_3)$

$p_3 = 3$

Multiplication (and division)

- $(v_1, p_1, n_1) * (v_2, p_2, n_2) = (v_3, p_3, n_3)$
- "Grade school multiplication"

$(v_1 = 5.25, p_1 = 5, n_1 = -3)$ - $(v_2 = 5, p_2 = 3, n_2 = -1)$

```
   0b101.01   0b10101.???
 * 0b101.??    0b0000.0??
 ------------   0b101.01?
 0b10101.???  +  0b??.???
  0b0000.0??  -----------
   0b101.01?  0b11010.01
    0b??.???  0b111??.
```

$p_3 = 3 \quad = \min(p_1, p_2)$

$v_3 = 28 \quad = \mathrm{round}(v_1 * v_2, p_3)$

$n_3 = 1$

Square root

- Only one operand, so no min or max
- $\mathrm{sqrt}(v_1, p_1, n_1) = (v_2, p_2, n_2)$

$(v_1 = 7.25, p_1 = 5, n_1 = -3)$

$p_2 = 6 \quad = p_1 + 1$

$v_2 = 2.6837 \quad = \mathrm{round}(\mathrm{sqrt}(v_1), p_3)$

$n_2 = -4$

```
sqrt(111.001) = 10.10101011…
sqrt(111.01?) = 10.10110001…
sqrt(111.011) = 10.10110111…
```

Upshot

- Sinking-point gives confidence that the bits in results are meaningful
- About as fast as IEEE 754
  - Only a few extra bits (log2($\mathbf{p}_{max}$) + 1, see paper)
  - Bitwise compatible for exact inputs
- Like IEEE 754, still an approximation
  - Not a sound guarantee
  - Does not replace other analyses, but helps indicate when to use them

Titanic

titanic.uwplse.org

Guaranteed to
float correctly

FPBench

fpbench.org

Common standards
for the floating-point
research community